Problem 88

The first atomic bomb was detonated on July 16, 1945, at the Trinity test site about 200 mi south of Los Alamos. In 1947, the U.S. government declassified a film reel of the explosion. From this film reel, British physicist G. I. Taylor was able to determine the rate at which the radius of the fireball from the blast grew. Using dimensional analysis, he was then able to deduce the amount of energy released in the explosion, which was a closely guarded secret at the time. Because of this, Taylor did not publish his results until 1950. This problem challenges you to recreate this famous calculation. (a) Using keen physical insight developed from years of experience, Taylor decided the radius r of the fireball should depend only on time since the explosion, t, the density of the air, ρ, and the energy of the initial explosion, E. Thus, he made the educated guess that $r = kE^a\rho^b t^c$ for some dimensionless constant k and some unknown exponents a, b, and c. Given that $[E] = \mathrm{ML^2T^{-2}}$, determine the values of the exponents necessary to make this equation dimensionally consistent. (Hint: Notice the equation implies that $k = rE^{-a}\rho^{-b}t^{-c}$ and that $[k] = 1$.) (b) By analyzing data from high-energy conventional explosives, Taylor found the formula he derived seemed to be valid as long as the constant k had the value 1.03. From the film reel, he was able to determine many values of r and the corresponding values of t. For example, he found that after 25.0 ms, the fireball had a radius of 130.0 m. Use these values, along with an average air density of 1.25 kg/m^3, to calculate the initial energy release of the Trinity detonation in joules (J). (*Hint*: To get energy in joules, you need to make sure all the numbers you substitute in are expressed in terms of SI base units.) (c) The energy released in large explosions is often cited in units of "tons of TNT" (abbreviated "t TNT"), where 1 t TNT is about 4.2 GJ. Convert your answer to (b) into kilotons of TNT (that is, kt TNT). Compare your answer with the quick-and-dirty estimate of 10 kt TNT made by physicist Enrico Fermi shortly after witnessing the explosion from what was thought to be a safe distance. (Reportedly, Fermi made his estimate by dropping some shredded bits of paper right before the remnants of the shock wave hit him and looked to see how far they were carried by it.)

Solution

Part (a)

Begin with Taylor's educated guess for the fireball's radius.

$$r = kE^a\rho^b t^c$$

Solve for k.

$$k = \frac{r}{E^a\rho^b t^c}$$

$$= rE^{-a}\rho^{-b}t^{-c}$$

Consider the dimensions of both sides.

$$[k] = \left[rE^{-a}\rho^{-b}t^{-c}\right]$$

$$= [r]\,[E]^{-a}\,[\rho]^{-b}\,[t]^{-c}$$

Plug in the dimensions of each variable: r is a distance, density is mass per unit volume, energy is mass times length squared divided by time squared, t is time, and k is dimensionless.

$$\begin{aligned} 1 &= [L]\left[\frac{ML^2}{T^2}\right]^{-a}\left[\frac{M}{L^3}\right]^{-b}[T]^{-c} \\ &= [L]^{1+2(-a)-3(-b)}[M]^{1(-a)+1(-b)}[T]^{-2(-a)+1(-c)} \\ &= [L]^{1-2a+3b}[M]^{-a-b}[T]^{2a-c} \end{aligned}$$

In order for this equation to be satisfied, all the exponents must be zero.

$$\left.\begin{aligned} 1-2a+3b &= 0 \\ -a-b &= 0 \\ 2a-c &= 0 \end{aligned}\right\}$$

Solving this system yields

$$a = \frac{1}{5} \quad \text{and} \quad b = -\frac{1}{5} \quad \text{and} \quad c = \frac{2}{5}.$$

Therefore, for the fireball radius to be dimensionally consistent, it has to be

$$r = kE^{1/5}\rho^{-1/5}t^{2/5}.$$

Part (b)

Set $k = 1.03$.

$$r = 1.03E^{1/5}\rho^{-1/5}t^{2/5}.$$

Solve this equation for E, the energy of the initial explosion.

$$E^{1/5} = \frac{r\rho^{1/5}}{1.03t^{2/5}}$$

$$\left(E^{1/5}\right)^5 = \left(\frac{r\rho^{1/5}}{1.03t^{2/5}}\right)^5$$

$$E = \frac{r^5\rho}{1.03^5t^2}$$

Plug in $r = 130.0$ m, $\rho = 1.25$ kg/m^3, and $t = 25$ ms $= 0.025$ s to determine the initial energy of the Trinity detonation in joules.

$$\begin{aligned} E &= \frac{(130.0\text{ m})^5\left(1.25\ \frac{\text{kg}}{\text{m}^3}\right)}{1.03^5(0.025\text{ s})^2} \\ &\approx 6.4\times10^{13}\text{ J} \end{aligned}$$

Part (c)

Convert the answer to part (b) into kilotons of TNT by multiplying by the appropriate conversion factors.

$$E = \frac{(130.0)^5\,(1.25)}{1.03^5(0.025)^2}\text{ J} \times \frac{1\text{ GJ}}{10^9\text{ J}} \times \frac{1\text{ t TNT}}{4.2\text{ GJ}} \times \frac{1\text{ kt TNT}}{1000\text{ t TNT}} \approx 15\text{ kt TNT}$$